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On a solution of the U(N) ⊃ O(N) state labelling problem for two-rowed representations

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Abstract. A discussion of the number of labelling operators specifying an abstract basis for an irreducible representation is given which indicates that in the U(N) ⊃ O(N) state labelling problem, for two-rowed irreducible representations $[p, q, 0, \ldots, 0]$ of U(N), a single additional commuting labelling operator Λ is required. An analysis of the tensor representations leads to a suitable labelling scheme involving an additional whole number label λ. A U(N) ⊃ O(N) branching theorem, for two-rowed representations, is formulated, and the branching multiplicities written down.

Combinatorial techniques developed by Green and Bracken in 1973 for the case of U(3) ⊃ SO(3), adapted to the case of U(N) ⊃ O(N), lead to a polynomial identity of the form

$$\mathscr{P}(\Lambda, \Phi, \ldots) = 0,$$

which implicitly defines the labelling operator Λ (with eigenvalue λ) in terms of certain O(N) invariants Φ (which are functions of the U(N) generators) and other known labelling operators and invariants. The calculations also give a cubic polynomial identity, satisfied by the $N \times N$ matrix of U(N) generators in two-rowed representations.

Some physical applications of the U(N) ⊃ O(N) state labelling problem are briefly mentioned.

1. Introduction

Many problems in different fields of physics can be viewed in a more abstract way as particular cases of certain general problems in group representation theory. Of this kind is the state labelling problem, in which it is required to find commuting labelling operators whose common eigenstates specify a basis for an irreducible representation of a group G, in such a way as to exhibit its irreducible contents as a representation of some subgroup, G^0. Such state labelling schemes must take account of the circumstance that a particular irreducible representation of G^0 may occur multiply within a given irreducible representation of G; the labels to be defined must distinguish between such equivalent representations, and remove the degeneracy. This is the case for the state labelling problem considered here, that of U(N), the group of $N \times N$ unitary matrices, and its subgroup O(N), the group of $N \times N$ orthogonal matrices.

The number of commuting operators required to provide a complete set of labels for abstract basis vectors in a representation of G is $\frac{1}{2}(n+m)$, where n is the dimension (number of real parameters) of G, and m its rank (Racah 1962). Hence, if n^0, m^0 are the quantities appropriate to the subgroup G^0, and an abstract basis is required in which the labels of G^0 are diagonal, then, taking account of the m labels characterizing the irreducible representation of G, a deficit of f labels, where

$$\tfrac{1}{2}(n+m) = m+f+\tfrac{1}{2}(n^0+m^0), \tag{1}$$

must be made good by the introduction of additional commuting labelling operators.

Irreducible representations of U(N) and of O(N) may be labelled $\mathrm{U}(N)[p_1, \ldots, p_N]$ and $\mathrm{O}(N)(l_1, \ldots, l_{[N/2]})$, respectively, where $N = 2[N/2]$, $2[N/2]+1$, according to whether N is even or odd, and where the p are all non-negative integers, and the l are either all integers or all half-integers, satisfying

$$p_1 \geqslant p_2 \geqslant \ldots \geqslant p_N \geqslant 0, \tag{2}$$

$$l_1 \geqslant l_2 \geqslant \ldots \geqslant l_{[N/2]} \geqslant 0. \tag{3}$$

The Gel'fand vectors (Gel'fand *et al* 1963) provide abstract bases for these representations, with the requisite numbers of labels, associated with irreducible representations of the subgroups in the chains:

$$\begin{aligned}&\mathrm{U}(N) \supset \mathrm{U}(N-1) \supset \ldots \supset \mathrm{U}(2) \supset \mathrm{U}(1),\\ &\mathrm{O}(N) \supset \mathrm{O}(N-1) \supset \ldots \supset \mathrm{O}(3) \supset \mathrm{SO}(2).\end{aligned} \tag{4}$$

The labels p and l are eigenvalues of corresponding hermitian labelling operators P and L, respectively.

The solution of the U(N) ⊃ O(N) state labelling problem requires on the other hand a non-canonical abstract basis involving additional labels $\lambda_1, \ldots, \lambda_f$ and commuting labelling operators $\Lambda_1, \ldots, \Lambda_f$, as well as the labels associated with the subgroup chain

$$\mathrm{U}(N) \supset \mathrm{O}(N) \supset \mathrm{O}(N-1) \supset \ldots \supset \mathrm{O}(3) \supset \mathrm{SO}(2), \tag{5}$$

in which the first link is degenerate. According to equation (1), the number f of such additional labels required is, for the general case of U(N) and O(N),

$$f = \tfrac{1}{4}(N^2 - N - 2[N/2]). \tag{6}$$

Of primary interest in the following will be a class of irreducible representations called here 'two rowed', that is, of the form $[p, q, 0, \ldots, 0] \equiv [p, q]$. The subduced irreducible representations of O(N) are also two rowed, of the form $(l, m, 0, \ldots, 0) \equiv (l, m)$. It follows from the form of the abstract basis vectors (Gel'fand *et al* 1963) that for this case, there are respectively $2N-1$ and $2N-4$ labels which do not vanish identically. The number f of additional labels required is therefore given by (compare equations (1) and (5))

$$(2N-1) = 2+f+(2N-4). \tag{7}$$

Thus, for two-rowed representations, the U(N) ⊃ O(N) state labelling problem requires just one additional commuting labelling operator Λ, with eigenvalue λ. The abstract basis vectors may be written

$$\left|\begin{matrix}p\\ q\end{matrix}\;\lambda\begin{pmatrix}l\\ m\end{pmatrix}\right\rangle, \tag{8}$$

where $\binom{l}{m}$ stands for the set of labels associated with the chain

$$\mathrm{O}(N) \supset \mathrm{O}(N-1) \supset \ldots \supset \mathrm{O}(3) \supset \mathrm{SO}(2),$$

in two-rowed representations.

The same situation thus obtains for the case in hand as for the U(3) $\supset$ SO(3) problem, where again there is just one missing label (Racah 1962). A theorem proved by Racah in the context of U(3) $\supset$ SO(3) gives important information about the operator Λ, if it be assumed valid for arbitrary N. The result is that, in an orthogonal labelling scheme, the diagonal matrix elements of certain independent SO(3) invariants, must be non-polynomial functions of λ. This is contradicted by the evaluation in § 4 of what is essentially the cubic invariant (compare Green and Bracken 1973). Hence, in the scheme here proposed, the operator Λ is non-hermitian, and the basis is non-orthogonal. The selection rule investigated by Racah (1962),

$$\Delta\lambda = 0, \pm 1, \tag{9}$$

as the simplest non-orthogonal case, is in fact satisfied in the scheme proposed (§ 4).

It should be mentioned that a solution of the U(N) $\supset$ O(N) state labelling problem, with appropriate modifications, also provides solutions of corresponding U(N) $\supset$ SO(N) and SU(N) $\supset$ SO(N) problems (the form taken in some applications). Thus, since the diagram

$$\begin{array}{ccc} \mathrm{U}(N) & \supset & \mathrm{O}(N) \\ \cup & \searrow & \cup \\ \mathrm{SU}(N) & \supset & \mathrm{SO}(N) \end{array} \tag{10}$$

commutes, in the sense of subduced representations, the U(N) $\supset$ O(N) state labelling scheme, here proposed for two-rowed representations, can be considered as a U(N) $\supset$ SO(N) state labelling scheme for the special class of N-rowed irreducible representations of the form $[p, q, s, \ldots, s]$, since

$$\mathrm{U}(N)[p, q, s, \ldots, s] \downarrow \mathrm{SO}(N) = \mathrm{U}(N)[p-s, q-s, 0, \ldots, 0] \downarrow \mathrm{SO}(N). \tag{11}$$

In particular, the complete U(3) $\supset$ SO(3) state labelling problem can be treated.

It is the intention in the following to show how, from an analysis of irreducible tensor representations, an abstract operational definition (albeit implicit) can be given of a suitable labelling operator Λ, in terms of known labelling operators and invariants constructed from the generators. The tensor representations are treated in § 2, and a simple U(N) $\supset$ O(N) branching law for two-rowed representations is found, together with the corresponding branching multiplicities. Certain O(N) invariants, constructed from the U(N) generators, are considered in § 3. These are evaluated in § 4 using combinatorial techniques developed by Green and Bracken (1973). A byproduct of this is a cubic polynomial identity, satisfied by the $N \times N$ matrix of generators of U(N) in two-rowed representations. Finally, it is shown how these evaluations are used in providing an implicit operational definition of Λ. Firstly, however, some physical applications of the U(N) $\supset$ O(N) state labelling problem are briefly mentioned.

The significance for physics of the U(3) $\supset$ SO(3) problem has engendered several studies (Green and Bracken (1973) provide references to earlier work). In particular Hughes (1973) and Judd *et al* (1974) propose orthogonal solutions, with eigenvalues of the additional label computed numerically, in contradistinction to the present scheme. The problem has arisen in various applications, in particle physics (Green and Bracken 1973), nuclear physics, and solid state physics.

The generalization considered here to arbitrary N includes the important case ($N = 4$) of U(3, 1) $\supset$ O(3, 1) (for the analysis of the finite-dimensional tensor representations, it is necessary only that the metric tensor be nonsingular; no particular signature need be specified). This can be applied in relativistic quantum mechanics (compare

Green and Bracken (1973) for the nonrelativistic case). It may also arise in relativistic elementary particle models, involving a nontrivial embedding of the Lorentz or Poincare group in a larger unitary group (whether or not a strict symmetry).

Certain state labelling problems in the classical groups are interrelated, so that a solution of one may provide indirectly solutions of related ones. The relationships between various embeddings and irreducible representations have been discussed for example by Quesne (1973).

2. The U(N) $\supset$ O(N) reduction via tensor representations

The space T^f of rank f tensors carries representations $\pi \to U_\pi$, $\gamma \to U_\gamma$, in general reducible, of S_f, the symmetric group on f symbols, and of GL(N, $\mathbb{C}$), the group of $N \times N$ nonsingular matrices, or of any of its subgroups.

It is well known (Hamermesh 1962) that the irreducible U(N) invariant subspaces of T^f are those of the form YT^f, where Y is a Young symmetry operator (symmetrizing over rows, and antisymmetrizing over columns) corresponding to some Young tableau.

Further reduction is possible for matrix subgroups leaving invariant a certain quadratic form g,

$$\gamma^{x_1'}{}_{x_1}\gamma^{x_2'}{}_{x_2}g_{x_1'x_2'} = g_{x_1x_2}. \tag{12}$$

(If g is symmetric, the group is O(N), or a pseudo-orthogonal group; if g is antisymmetric, the group is Sp(N).) The corresponding irreducible invariant subspaces are those of the form $Y\bar{T}^f$, where $\bar{T}^f$ is the space of completely traceless rank f tensors (Hamermesh 1962).

In particular, the irreducible representation U(N)$[p, q]$ is realized on the space $T[p, q]$ of tensors of symmetry type $\{p, q\}$. Tensor components are contravariant, and are written

$$\begin{bmatrix} s_1 s_2 \dots s_q\, u_1 \dots u_r \\ t_1 \quad \dots t_q \end{bmatrix}, \quad r = p - q, \tag{13}$$

and are completely antisymmetric within each column, completely symmetric with respect to interchanges of columns of equal length, and possess in addition the symmetries

$$\begin{aligned} \begin{bmatrix} s_1 s_2 \\ t_1 t_2 \end{bmatrix} &= \begin{bmatrix} s_1 t_1 \\ s_2 t_2 \end{bmatrix} - \begin{bmatrix} t_1 s_1 \\ s_2 t_2 \end{bmatrix}, \\ \begin{bmatrix} s_1 u_1 \\ t_1 \end{bmatrix} &= \begin{bmatrix} s_1 t_1 \\ u_1 \end{bmatrix} - \begin{bmatrix} t_1 s_1 \\ u_1 \end{bmatrix}, \end{aligned} \tag{14}$$

in each pair of columns.

Similarly, the irreducible representation O(N)(l, m) is realized on the space $T(l, m)$ of completely traceless tensors of symmetry type $\{l, m\}$.

The reduction of an irreducible representation of U(N) with respect to O(N) can be considered in two stages: firstly, the decomposition into traceless parts, and secondly, the application of Young operators to each of these traceless parts. The first stage is dealt with in the following.

Theorem 1. Every tensor $t \in T^f$ can be uniquely decomposed into two summands,

$$t = \bar{t} + u, \tag{15}$$

where $\bar{t}$ is completely traceless, and u has the form

$$u^{x_1 \ldots x_f} = g^{x_1 x_2} u_{(12)}^{x_3 \ldots x_f} + \ldots, \tag{16}$$

with $\frac{1}{2}f(f-1)$ summands. $\bar{t}$ and u are orthogonal, since

$$\bar{t}^{x_1 \ldots x_f} u_{x_1 \ldots x_f} = 0.$$

Proof. Weyl (1939, p 150).

It follows from equation (15) that each summand of u can be expressed as a linear combination of traces of t. By repeating the procedure a decomposition of t into a sum of products of g with completely traceless tensors is obtained:

$$t = \sum_{\pi,\tau} \xi_{\pi\tau} U_\pi(g_\tau \bar{t}_\tau), \tag{17}$$

where the ξ are scalars, the $\bar{t}_\tau$ are traceless parts of traces t_τ of t, the g_τ are products of g's, and the summation extends over all permutations π and traces τ.

The application of Young operators to each of the $\bar{t}_\tau$ produces a sum of irreducible tensors. The various distinct tensors obtained after grouping terms are the unique irreducible O(N) constituents. Equivalent O(N) constituents which may occur will be distinguished by their arising from different traces of the original tensor.

For the special case of completely symmetric tensors, the above considerations lead to the familiar result that a completely symmetric rank f tensor yields angular momenta $f, f-2, f-4, \ldots$. For tensors of two-rowed symmetry type, the reduction according to the above scheme can be carried out explicitly in simple cases. For example, for type $\{3, 2\}$,

$$\begin{aligned}
\begin{bmatrix} pqr \\ st \end{bmatrix} &= \begin{bmatrix} \overline{pqr} \\ st \end{bmatrix} - \xi_{(30)} Y\left[g^{st} \begin{bmatrix} \overline{pqr} \\ \cdot\,\cdot \end{bmatrix} \right] - \xi_{(21)} Y\left[g^{tr} \begin{bmatrix} \overline{pq\cdot} \\ s\cdot \end{bmatrix} \right] - \xi_{(10)} Y\left[g^{pq} g^{st} \begin{bmatrix} \overline{\cdot\,\cdot\, r} \\ \cdot\,\cdot \end{bmatrix} \right], \\
\begin{bmatrix} \overline{pqr} \\ \cdot \end{bmatrix} &= \begin{bmatrix} pqr \\ ii \end{bmatrix} + \begin{bmatrix} qrp \\ ii \end{bmatrix} + \begin{bmatrix} rpq \\ ii \end{bmatrix} - \frac{2}{N+2}\left(g^{qr} \begin{bmatrix} iip \\ jj \end{bmatrix} + g^{rp} \begin{bmatrix} iiq \\ jj \end{bmatrix} + g^{pq} \begin{bmatrix} iir \\ jj \end{bmatrix} \right), \\
\begin{bmatrix} \overline{pq\cdot} \\ s\cdot \end{bmatrix} &= \begin{bmatrix} pqi \\ si \end{bmatrix} + \frac{1}{2(N-1)}\left(g^{sq} \begin{bmatrix} iip \\ jj \end{bmatrix} - g^{pq} \begin{bmatrix} iis \\ jj \end{bmatrix} \right), \\
\begin{bmatrix} \overline{\cdot\,\cdot\, r} \\ \cdot\,\cdot \end{bmatrix} &= \begin{bmatrix} iir \\ jj \end{bmatrix}
\end{aligned} \tag{18}$$

$$\xi_{(30)} = \frac{1}{3(N-2)}, \qquad \xi_{(21)} = \frac{1}{3(N+1)}, \qquad \xi_{(10)} = \frac{1}{2(N-1)(N+2)}$$

where the bar indicates symmetrized, completely traceless tensors, where $[ii] \equiv [ij] g_{ij}$, and $Y \equiv Y\{3, 2\}$, yielding for example

$$Y[g^{st}[pqr]] = g^{st}[pqr] - g^{pt}[sqr] - g^{sq}[ptr] + g^{pq}[str].$$

Equations (18) show that the irreducible O(N) constituents are given by

$$\mathrm{U}(N)[3,2] \downarrow \mathrm{O}(N) = (3,2) \oplus (3,0) \oplus (2,1) \oplus (1,0).$$

The above example shows that the irreducible O(N) constituents are in one-to-one correspondence with certain standard traces of the U(N) tensor. The form of the standard trace in the general case is given by the following.

Theorem 2. An arbitrary trace of a tensor of symmetry type $\{p, q\}$ can be expressed in terms of traces in standard form, labelled by whole numbers (κ, λ, μ), with graphical representation as in figure 1, where

$$\begin{aligned} &0 \leqslant 2\lambda \leqslant p, \qquad 0 \leqslant 2\mu \leqslant q, \qquad \lambda \geqslant \mu; \\ &0 \leqslant \kappa \leqslant \min(q-2\mu, p-2\lambda) - \max(q-2\lambda, 0); \end{aligned} \tag{19}$$

$$\begin{aligned} &(q_1, q_2, q_3, r_1, r_2) = (2\mu, m, (q-2\mu)-m, m-(q-2\lambda), (p-2\lambda)-m); \\ &m = \kappa + \max(q-2\lambda, 0). \end{aligned} \tag{20}$$

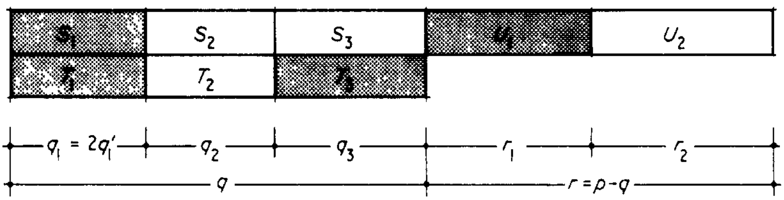

Figure 1. Standard trace for symmetry type $\{p, q\}$.

Shaded portions of the figure represent contractions (of adjacent indices), and unshaded portions represent free indices. $(\lambda+\mu)$ is the total number of pairs contracted, μ the number of 2×2 blocks contracted, and κ describes the location of the remaining $(\lambda-\mu)$ contractions.

Proof. By induction, using equations (14). The requirement $q_2 \geqslant 0$ for fixed λ and μ determines the range of κ.

Each standard trace (κ, λ, μ) can be further decomposed into a sum of symmetrized standard traces by the application of all possible Young operators $Y\{s_1, s_2\}$. However equations (14) imply that Y annihilates (κ, λ, μ) for $s_2 < q_2$, and for $s_2 > q_2$ produces a symmetrized $(\kappa', \lambda', \mu')$ with different values of the labels. Hence the distinct symmetrized standard traces are just the symmetrized (κ, λ, μ).

Now to each symmetrized standard trace (κ, λ, μ) there corresponds by theorem 1 a completely traceless tensor of the same symmetry type $\{q_2+q_3+r_2, q_2\}$, which therefore belongs to an irreducible representation of O(N). The irreducible O(N) constituents of U(N)$[p, q]$ may therefore be labelled by (κ, λ, μ), or by $(\lambda; l, m)$, by rearranging equations (20), where

$$\begin{aligned} &l = q_2+q_3+r_2 = (p+q) - 2(\lambda+\mu) - \kappa - \max(q-2\lambda, 0), \\ &m = q_2 = \kappa + \max(q-2\lambda, 0); \\ &q_1+q_2+q_3 = q, \\ &r_1+r_2 = r = p-q; \end{aligned} \tag{21}$$

$$(q_1, q_2, q_3, r_1, r_2) = ((p+q)-(l+m)-2\lambda, m, l-(p-2\lambda), m-(q-2\lambda), (p-2\lambda)-m). \tag{22}$$

By rearranging equations (19), the following $U(N) \supset O(N)$ branching theorem for two-rowed representations is deduced.

Theorem 3.

$$N \geqslant 4: \qquad [p, q] \downarrow O(N) = \sum_{D_N} \oplus (\lambda; lm)$$

$$N = 3: \qquad [p, q] \downarrow O(3) = \left(\sum_{D_3} \oplus (\lambda; l)\right) \oplus \left(\sum_{D_3^*} \oplus (\lambda; l)^*\right)$$

$$N = 2: \qquad [p, q] \downarrow O(2) = \left(\sum_{D_2} \oplus (\lambda; l)\right) \oplus \left(\sum_{D_2^*} \oplus (\lambda; l)^*\right) \oplus \left(\sum_{D_2^{0*}} \oplus (\lambda; 0)^*\right) \tag{23}$$

$$\begin{aligned} D_N: \quad & 0 \leqslant 2\lambda \leqslant p; \qquad (p+q)-(l+m) \text{ even}; \qquad m \leqslant l; \\ & \max(q-2\lambda, 0) \leqslant (l+m)-(p-2\lambda) \leqslant q, \\ & \max(q-2\lambda, 0) \leqslant m \leqslant \min((l+m)-(p-2\lambda), p-2\lambda); \\ & D_3 = D_3(m = 0); \qquad D_3^* = D_3(m = 1); \\ & D_2 = D_2(m = 0); \qquad D_2^* = D_2(m = 2); \qquad D_2^{0*} = D_2(l = m = 1). \end{aligned} \tag{24}$$

Here $(l)^*$ is the representation associate to (l) (differing by the alternating character). The modification rules of Murnaghan (1949, p 282) have been used for $N = 2$ and $N = 3$.

The $U(N) \supset SO(N)$ branching theorem for two-rowed representations is similar to the above, except that the associated representations become identical, and (Boerner 1963)

$$\begin{aligned} O(4)(l, m) \downarrow SO(4) &= (l, m) \oplus (l, -m) \qquad (m \neq 0), \\ O(2)(l) \downarrow SO(2) &= (l) \oplus (-l) \qquad (l \neq 0). \end{aligned} \tag{25}$$

That the label λ actually distinguishes equivalent $O(N)$ representations is clear from the fact that for fixed λ and (l, m), $(l+m, m)$ lies in D_N^λ at most once. The multiplicity of $O(N)(l, m)$ within $U(N)[p, q]$ is obtained from the range of λ (equations (24)):

$$\max(p-l, q-m) \leqslant 2\lambda \leqslant \min(p-m, (p+q)-(l+m)). \tag{26}$$

The remaining sections aim to construct an abstract operational definition of a labelling operator Λ with eigenvalues corresponding to the label λ, introduced here in the context of tensor representations.

3. U(*N*) and O(*N*) invariants

The single additional labelling operator Λ which is to be introduced in order to complete the state labelling scheme must be an $O(N)$ invariant, and will therefore be defined in terms of a complete set of $O(N)$ invariants. These are now considered.

The generators of $U(N)$ are operators $a^i{}_j$ having the commutation relations (Green 1971)

$$[a^i{}_j, a^k{}_l] = \delta^i{}_l a^k{}_j - \delta^k{}_j a^i{}_l, \qquad 1 \leqslant i, j, k, l \leqslant N. \tag{27}$$

The operators α_{ij}, defined by

$$\alpha_{ij} = a_{ij} - \bar{a}_{ij},$$
$$\bar{a}_{ij} = a_{ji} = a^k{}_i g_{jk}, \tag{28}$$

are the generators of the subgroup O(N) in the representation, and have the commutation relations (Green 1971)

$$[\alpha_{ij}, \alpha_{kl}] = g_{kj}\alpha_{il} - g_{il}\alpha_{kj} - g_{ki}\alpha_{jl} + g_{jl}\alpha_{ki}. \tag{29}$$

The $a^i{}_j$, $\alpha^i{}_j$ can be considered as $N \times N$ matrices, and matrix products and traces can be defined as for c numbers (Bracken and Green 1971):

$$(a^n)^i{}_j = (a^{n-1})^i{}_k a^k{}_j,$$
$$\langle a \rangle = a^i{}_i. \tag{30}$$

It follows from the commutation relations that, in addition to the invariants $\langle \alpha^n \rangle$ constructed from the generators of O(N) (Bracken and Green 1971), traces of the general form $\langle a^{n_1}\bar{a}^{n_2}\ldots\rangle$, constructed from the generators of U(N), are O(N) invariants (in particular, the U(N) invariants $\langle a^n \rangle$ are of this form).

For two-rowed representations, it suffices to consider the eight cubic, and sixteen quartic, O(N) invariants of this form. These are of course not all independent. The commutation relations themselves imply certain inter-relationships; for example it can be proved generally that

$$\langle a^{n_1}\bar{a}^{n_2}\rangle = \langle \bar{a}^{n_2}a^{n_1}\rangle. \tag{31}$$

In addition, the invariants $\langle a^n \rangle$ and $\langle \alpha^n \rangle$ can be expressed in terms of the labels $[p, q]$ and (l, m) respectively. The expansion of $\langle \alpha^n \rangle = \langle (a-\bar{a})^n \rangle$ then imposes further restrictions.

Finally, the cubic polynomial identity, equations (42), satisfied by the matrix of U(N) generators in two-rowed representations, can be used to evaluate directly any of the O(N) invariants involving products of a^3 or $\bar{a}^3$.

The result of the above considerations is that, for two-rowed irreducible representations of U(N), there is just one independent additional cubic, and one quartic, O(N) invariant. These can be taken to be $\frac{1}{2}\langle a\bar{a}a + \bar{a}a\bar{a}\rangle$ and $\frac{1}{2}\langle a\bar{a}a\bar{a} + \bar{a}a\bar{a}a\rangle$, respectively. It is in terms of these invariants that the additional labelling operator Λ is to be defined.

4. Evaluation of invariants

The tensor reduction

$$T[p, q] = \sum_{D_N} \oplus\, T(\lambda\,; l, m) \tag{32}$$

carried out in § 2 showed each irreducible constituent corresponding to a standard trace $(\lambda\,; l, m)$ to be a linear combination of tensors of the form

$$\Gamma^0_{(lm)} t_{(\lambda;lm)} = Y(g_{(\lambda;lm)}\Gamma_{(lm)} t_{(\lambda;lm)}), \tag{33}$$

where $\Gamma_{(lm)}$ is an operator effecting the projection on to the traceless tensor corresponding to $t_{(\lambda;lm)}$ (equation (15)), $g_{(\lambda;lm)}$ is an appropriate product of g's, and the Young operator $Y\{p, q\}$ appears because of the uniqueness of the decomposition of equation (15).

Shift operators $\Lambda^{\pm}$ which change the value of λ are defined on $T(\lambda; l, m)$ by their action on tensors of this form:

$$\Lambda^{\pm}(\Gamma^{0}_{(lm)}t_{(\lambda;lm)}) = \Gamma^{0}_{(lm)}t_{(\lambda\pm 1;lm)} \qquad (0 \leqslant 2(\lambda \pm 1) \leqslant p),$$
$$\Lambda^{\pm}(\Gamma^{0}_{(lm)}t_{(\lambda;lm)}) = 0 \qquad \text{(otherwise)}. \tag{34}$$

If Φ is an arbitrary O(N) invariant in the representation carried by $T[p, q]$, then a modified invariant Φ^0 is defined on $T[p, q] = \Sigma_{D_N} \oplus T(\lambda; l, m)$ by

$$\Phi^0 t = \Gamma^{0}_{(lm)}(\Phi t)_{(\lambda;lm)}. \tag{35}$$

This follows from the structure of equation (33) and the properties of g and Y under orthogonal transformations.

The Φ^0, rather than the Φ, are more readily evaluated in the present framework. The invariance of Φ^0 means that $\Phi^0 T(\lambda; lm)$ is an invariant subspace. Moreover Φ^0 can only mix the label λ, leaving the other O(N) labels invariant. Hence

$$\Phi^0 : T(\lambda; l, m) \rightarrow \sum_{D_N} \oplus\ T(\lambda'; l, m). \tag{36}$$

In the calculations discussed below, the traces $(\Phi t)_{(\lambda;lm)}$ are evaluated. In general it is found that

$$(\Phi t)_{(\lambda;lm)} = \xi U_{\pi} t_{(\lambda;lm)} + \xi^{+} U_{\pi^{+}} t_{(\lambda+1;lm)} + \xi^{-} U_{\pi^{-}} t_{(\lambda-1;lm)}, \tag{37a}$$

where the scalars ξ, $\xi^{\pm}$ are polynomials in N, p, q, λ, l and m, and where $\pi, \pi^{\pm} \in S_{l+m}$ act on indices which are symmetrized by the application of $\Gamma_{(lm)}$. Hence, using equations (33), (34) and (35),

$$\Phi^0 = \xi + \xi^{+}\Lambda^{+} + \xi^{-}\Lambda^{-}. \tag{37b}$$

If on the other hand the cases

$$(\Phi t)_{(\lambda;lm)} = \xi U_{\pi} t_{(\lambda;lm)} \tag{38a}$$
$$(\Phi t)_{(\lambda;lm)} = \xi g_{\sigma} t_{(\lambda;lm)\sigma} \tag{38b}$$

occur, where σ is a trace disjoint from $(\lambda; lm)$, and U_{π} antisymmetrizes over indices symmetrized by $\Gamma_{(lm)}$, then $\Phi^0 t$ vanishes (in the second case because the traceless part of any product of g's is zero).

Of course, if Φ is an invariant of U(N), or is a function only of the O(N) generators, then only the first term of equation (37a) appears. Thus, for the invariants $\langle a \rangle$, $\langle a^2 \rangle$, $\langle a^3 \rangle$, $\langle \bar{a}^3 \rangle$ and $\langle a\bar{a} \rangle$, a direct evaluation gives

$$\langle a \rangle = \langle \bar{a} \rangle = p + q,$$
$$\langle a^2 \rangle = \langle \bar{a}^2 \rangle = p(p+N-1) + q(q+N-3),$$
$$\langle a^3 \rangle = (2N-3)\langle a^2 \rangle + \tfrac{1}{2}[3\langle a^2 \rangle - \langle a \rangle^2 - (3N-5)\langle a \rangle - 2(N-1)(N-2)]\langle a \rangle,$$
$$\langle \bar{a}^3 \rangle = (N-3)\langle a^2 \rangle + \tfrac{1}{2}[3\langle a^2 \rangle - \langle a \rangle^2 - (3N-7)\langle a \rangle - 2(N-1)(N-2)]\langle a \rangle,$$
$$\langle a\bar{a} \rangle = p(p+N-1) + q(q+N-3) - l(l+N-2) - m(m+N-4), \tag{39}$$

the last equation being in accord with the identity (Bracken and Green 1971)

$$\langle a\bar{a} \rangle = \langle a^2 \rangle - \tfrac{1}{2}\langle \alpha^2 \rangle. \tag{40}$$

In fact it proves possible to evaluate a^3 and $\bar{a}^3$ directly. Bracken and Green (1971) have given a polynomial identity of degree N satisfied by the $N \times N$ matrix of $U(N)$ generators in the irreducible representation $[p_1, \ldots, p_N]$. For $[p, q, 0, \ldots, 0]$ this reduces to

$$\begin{aligned}
&(a-0)(a-0-1)(\ldots)(a-0-N+3)(a-q-N+2)(a-p-N+1) = 0, \\
&(\bar{a}-0+N-1)(\bar{a}-0+N-2)(\ldots)(\bar{a}-0+2)(\bar{a}-q+1)(\bar{a}-p+0) = 0, \qquad (41)
\end{aligned}$$

while the result of the direct evaluation is a cubic polynomial identity of degree N. For $[p, q, 0, \ldots, 0]$, and $[p, p, 0, \ldots, 0]$, respectively,

$$\begin{aligned}
&a(a-q-N+2)(a-p-N+1) = 0, \\
&(\bar{a}+2)(\bar{a}-q+1)(\bar{a}-p) = 0, \\
a^3 &= (\langle a \rangle + 2N-3)a^2 + \tfrac{1}{2}[\langle a^2 \rangle - \langle a \rangle^2 - (3N-5)\langle a \rangle - 2(N-1)(N-2)]a, \\
\bar{a}^3 &= (\langle a \rangle - 3)\bar{a}^2 + \tfrac{1}{2}[\langle a^2 \rangle - \langle a \rangle^2 - (N-7)\langle a \rangle - 4]\bar{a} + (\langle a^2 \rangle - \langle a \rangle^2 - (N-3)\langle a \rangle); \\
&a(a-p-N+2) = 0, \\
&(\bar{a}+2)(\bar{a}-p) = 0. \qquad (42)
\end{aligned}$$

The calculations are all carried out using equations (35), (37) and (38), using the standard trace $(\lambda; l, m)$ of figure 1 and the symmetry properties, equation (14). The action of the generators of $U(N)$ in tensor representations is given by (Green 1971):

$$(a^i{}_j t)^{x_1 \ldots x_f} = \delta^{x_1}{}_j t^{i x_2 \ldots x_f} + \ldots + \delta^{x_f}{}_j t^{x_1 \ldots x_{f-1} i}. \qquad (43)$$

Such tensor substitutions, and more complicated combinations, can be represented (Green and Bracken 1973):

$$\begin{aligned}
&1 = (\quad), \\
&a^i{}_j = \delta^x{}_j \binom{i}{x}, \\
&a^i{}_j a^k{}_l = \delta^{x'}{}_j \delta^x{}_l \binom{i\ \ k}{x'\,x} + \delta^k{}_j \delta^x{}_l \binom{i}{x}, \qquad x' \neq x, \qquad (44)
\end{aligned}$$

where the index label x runs over all indices $x_1, \ldots, x_f$ of the index set, and where the bracket $\binom{i}{x}$ indicates that for each of these values, the label i is to be substituted in the appropriate location.

As an example of these calculations, the $O(N)$ invariant $\langle a\bar{a} \rangle$ is evaluated on the irreducible representation $O(N)(l, 0)$ contained within $U(N)[p, 0]$. The corresponding tensors are completely symmetrical. The associated standard trace is represented graphically in figure 2 (compare figure 1), where shaded portions represent contractions, and

$$\begin{aligned}
n_1 &= l, \\
n_2 &= 2n_2' = p - l.
\end{aligned}$$

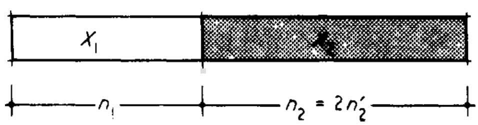

Figure 2. Standard trace for symmetry type $\{p, 0\}$.

From equation (44),

$$\langle a\bar{a}\rangle = g_{ij}a^i{}_k a^j{}_l g^{kl} = \delta^x{}_i\binom{i}{x} + g^{xx'}\binom{i\,j}{xx'}g_{ij},$$
$$\langle a\bar{a}\rangle = \langle a\rangle + (XX);$$
$$g^{xx'}\binom{i\,j}{xx'}g_{ij} = (XX) = (X_1X_1)+(X_2X_2)+(X_1X_2), \tag{45}$$

the terms in equation (45) indicating restrictions of the index summation. It suffices to write down

$$(X_1X_1)[x_1x_2] = 2g^{x_1x_2}[ll],$$
$$(X_1X_2)[x_1x_2i_1i_2] = 2g^{x_1i_1}[lx_2li_2]+2g^{x_1i_2}[lx_2i_1l]+2g^{x_2i_1}[x_1lli_2]+2g^{x_2i_2}[x_1li_1l], \tag{46}$$

where the repeated index stands for contraction with the covariant metric tensor, $[ll] = [ij]g_{ij}$. Hence

$$(X_1X_1)[x_1x_2] \to 0, \qquad (X_1X_1) \to 0 \qquad \text{(equation (38}b\text{))};$$
$$(X_2X_2)[i_1i_2] \to 2N[ll](n_2' = 1), \qquad (X_2X_2)[i_1i_2i_3i_4] \to (4N+8)[llmm](n_2' = 2),$$
$$(X_2X_2) \to n_2(n_2+N-2) \qquad \text{(equation (37}a\text{))};$$
$$(X_1X_2)[x_1x_2i_1i_2] \to 8[x_1x_2ll](n_1 = 2, n_2' = 1),$$
$$(X_1X_2) \to 2n_1n_2 \qquad \text{(equation (37}a\text{))}; \tag{47}$$

where $\to$ indicates the result after applying the standard trace, and equations (37) and (38). Adding, using $\langle a\rangle = p$,

$$(XX) = n_2^2+2n_1n_2+(N-2)n_2,$$
$$\langle a\bar{a}\rangle = p(p+N-1)-l(l+N-2), \tag{48}$$

in agreement with equation (39).

Similar combinatorial techniques can be used to evaluate the cubic O(N) invariant $\frac{1}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle$ of § 3:

$$\tfrac{1}{2}\langle a\bar{a}a+\bar{a}a\bar{a}\rangle = \Phi_0+(q+\tfrac{1}{2}N-2)\langle a\bar{a}\rangle+(p-q+1)\Phi_1+(p+q-2)\Phi_2+\Phi_3, \tag{49}$$
$$\Phi_0 = (p-q)(p+1)+\tfrac{1}{2}(p+q)^2,$$
$$\Phi_1 = g^{uu'}\binom{i\,j}{uu'}g_{ij}, \qquad u \neq u',$$
$$\Phi_2 = \left(g^{su}\binom{ij}{su}+g^{tu}\binom{ij}{tu}\right)g_{ij},$$
$$\Phi_3 = \left(g^{ss'}\binom{i\,u\,j}{uss'}+g^{st}\binom{i\,u\,j}{ust}+g^{ts}\binom{i\,u\,j}{uts}+g^{su}\binom{i\;u'j}{u's\;u}+g^{tu}\binom{i\;u'j}{u't\;u}+g^{tt'}\binom{i\,u\,j}{utt'}\right)g_{ij},$$

and after a lengthy calculation, it is found that

$$\Phi_1 = r_1^2+2r_1r_2+(N-2)r_1, \tag{50}$$
$$\Phi_2 = q_1r_1+q_2r_1+q_3r_1+q_1r_2,$$
$$\begin{aligned}\Phi_3 = {} & q_1^2r_1+q_1^2r_2+q_3^2r_1+q_1r_1^2+q_2r_1^2+q_3r_1^2+2q_1q_2r_1+2q_1q_2r_2+2q_1q_3r_1\\ & +2q_1q_3r_2+2q_2q_3r_1+2q_1r_1r_2+q_2r_1r_2+q_3r_1r_2+(N-3)q_1r_1\\ & +(N-3)q_1r_2+(N-4)q_3r_1+2\Lambda^+(q_1r_2^2-q_1r_2)\\ & -\tfrac{1}{2}\Lambda^-(q_3^2r_1+2q_2q_3r_1+(N-4)q_3r_1),\end{aligned}$$

using the parameters of equation (22). Details of all the above calculations are given by Jarvis (1974).

Absorbing the shift-operator independent term on the left-hand side, equations (49) and (50) can be written

$$\Phi = \Lambda^+\xi^+(p,q;\lambda;l,m)+\Lambda^-\xi^-(p,q;\lambda;l,m), \tag{51}$$

$$\xi^+ = 2(p+q-2\lambda-l-m)(p-2\lambda-m)(p-2\lambda-m-1),$$
$$\xi^- = \tfrac{1}{2}(p-2\lambda-l)(q-2\lambda-m)(p-2\lambda-l-2m-N+4). \tag{52}$$

By commuting equation (51) with Λ, and using $[\Lambda,\Lambda^\pm] = \pm\Lambda^\pm$,

$$[\Lambda,\Phi] = \Lambda^+\xi^+(p,q;\lambda;l,m)-\Lambda^-\xi^-(p,q;\lambda;l,m), \tag{53}$$

and by combining equations (51) and (53),

$$4\Lambda^+\Lambda^-\xi^+\xi^- +([\Lambda,\Phi]+\Phi)([\Lambda,\Phi]-\Phi) = 0. \tag{54}$$

However, from equations (24) and (52), the factor $\xi^+\xi^-$ vanishes for extreme values of λ. It follows that equation (54) may be replaced by

$$4\xi^+\xi^-(P,Q;\Lambda;L,M)+([\Lambda,\Phi]+\Phi)([\Lambda,\Phi]-\Phi) = 0. \tag{55}$$

A similar procedure could be carried out if, rather than equation (53), there were a second independent equation of the same form, such as would be afforded by the evaluation of the independent quartic O(N) invariant $\frac{1}{2}\langle a\bar{a}a\bar{a}+\bar{a}a\bar{a}a\rangle$ of § 3 (compare Green and Bracken 1973), leading to an equation similar to equation (55).

Equation (55) provides the desired implicit operational definition

$$\mathscr{P}(\Lambda,\Phi;P,Q;L,M) = 0, \tag{56}$$

of Λ in terms of known labelling operators and invariants, where $\mathscr{P}$ is a polynomial. It should be pointed out, however, that since a definite choice of normalization of the abstract basis has not been imposed, the Λ in the definition should possibly be accompanied by a further factor polynomial in the labels. This could also be accomplished by a subsequent redefinition of the un-normalized Λ.

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